\textbf{CADO-NFS, a Number Field Sieve implementation}

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Motivations

Integer factorization \((N = pq \rightarrow \text{find } p, q)\) is a hard problem.

- Pre-1980’s: a stumbling block in mathematical computations, and a challenging problem. Some significant advances in the 1970’s.

- 1978-present: IF has attracted considerable attention because of its relevance for cryptography through the RSA cryptosystem.
CADO-nfs: an implementation of NFS

The fastest integer factoring algorithm is the Number Field Sieve.

- Very complicated algorithm. Embarks lots of number theory.
  (much more involved than, e.g., the ECM factoring algorithm)
- Very few available implementations. State of the art is at best
  bits and pieces from here and there.

CADO project. Write our own code. Joint effort, started in 2007.

- Actively developed. Playground for new ideas.
- Certainly beatable, but contains nice algorithms.
- No refrain to reorganizing the code to (changing) taste every
  so often.

CADO-nfs is LGPL, and written (almost) entirely in C. To date,
~ 120 kLOC.
Objectives for an NFS program

An NFS program like CADO-NFS can be used for various purposes.

- « below-NFS-threshold » numbers. Below 120dd, QS is faster. ⇒ intended for routine checking, timings are not the issue.
- Numbers which explore the limitations of the current code. Do growing sizes, add optimizations. Ongoing effort. Currently doing 700 bits.
- Record-size numbers. CADO-NFS can’t factor rsa768, but participating to rsa768 taught us a lot.

Note: CADO-NFS is clearly not an integrated factoring machinery. CADO-NFS does not include ECM, QS, ...

- No interaction with a user.
- Interface: a collection of programs driven by a main script.
The feasibility limit explored by NFSrecords is used to determine key sizes for RSA.

- **SSL/TLS.** **CA root certificates** are installed by default in browsers.
  - Linux laptop, 2005: 1024b (50%), 2048b (48%), 4096b (2%);
  - Linux laptop, 2009: 1024b (31%), 2048b (58%), 4096b (10%).

- **EMV credit cards (a.k.a. chip and pin).**
  - Most chip public keys are 960b. Some 1024b (until end of 2009, some had a 896b key).

**Factoring experiments:** decision-driving data for setting key sizes.
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The GNFS setup

For factoring “general” $N$, GNFS uses:

- a number field $K = \mathbb{Q}(\alpha)$ defined by $f(\alpha) = 0$, for $f$ irreducible over $\mathbb{Q}$ and $\deg f = d$;

- Another irreducible polynomial $g$ such that $f$ and $g$ have a common root $m \mod N$ (example: $g = x - m$).

$g$ defines the rational side, $f$ defines the algebraic side.

Choosing $f$ and $g$ is referred to as the polynomial selection step.

General plan: Obtain relations, and combine them to obtain:

$$x^2 \equiv y^2 \mod N.$$
Relations in NFS

\[
\begin{align*}
\mathbb{Z}[x] & \quad \psi^{(1)} : x \rightarrow m \\
\mathbb{Z}[m] & \quad \psi^{(2)} : x \rightarrow \alpha \\
\mathbb{Z}/N\mathbb{Z} & \quad \varphi^{(1)} : t \rightarrow t \mod N \\
\mathbb{Z} & \quad \varphi^{(2)} : \alpha \rightarrow m \mod N
\end{align*}
\]

Take for example \( a - bx \) in \( \mathbb{Z}[x] \). Suppose for a moment that:

- the integer \( a - bm \) is smooth: product of factor base primes;
- the algebraic integer \( a - b\alpha \) is also a product.

Then we have an multiplicative relation in \( \mathbb{Z}/N\mathbb{Z} \). We can hope to combine many such relations to form a congruence of squares.

\[
R = (a_1 - b_1 m) \times \cdots \times (a_k - b_k m) = \Box,
\]

\[
A = (a_1 - b_1 \alpha) \times \cdots \times (a_k - b_k \alpha) = \Box,
\]

\[
\varphi^{(1)}(R) \equiv \varphi^{(2)}(R) \mod N.
\]
Recognizing when $a - b\alpha$ factors

**Major obstruction:** $\mathbb{Z}[\alpha]$ not a UFD. “Factoring” $(a - b\alpha)$ won’t work too well.

The proper object to look at is the factorization of the principal **ideal** generated by $(a - b\alpha)$ in the ring of integers of $K$.

- Some obstructions (ramifications, who’s the maximal order) must be worked around.
- Essentially, we want the integer

$$\text{Norm}_{K/\mathbb{Q}}(a - b\alpha) = \text{Res}(a - bx, f) = b^df(a/b) = F(a, b)$$

to be smooth. Nothing terribly complicated.
Complexity of NFS

For factoring an integer $N$, GNFS takes time:

$$L_N[1/3, (64/9)^{1/3}] = \exp \left( (1 + o(1))(64/9)^{1/3}(\log N)^{1/3}(\log \log N)^{2/3} \right).$$

This is sub-exponential.

Note: some special numbers allow for a faster variant NFS, with complexity

$$L_N[1/3, (32/9)^{1/3}] = \exp \left( (1 + o(1))(32/9)^{1/3}(\log N)^{1/3}(\log \log N)^{2/3} \right).$$
NFS might not be the simplest algorithm on earth, but:

- obstructions have been dealt with already long ago. See literature.
- the bottom line is simple: everything boils down to assembly/C/MPI.

**Polynomial selection:** find \( f, g \);

**Sieving:** find many \( a, b \) s.t. \( F(a, b) = b^d f(a/b) \) and \( G(a, b) \) smooth.

**Linear algebra:** combine \( a, b \) pairs to get a congruence of squares.

\[ \Rightarrow \] solve a large sparse linear system over \( \mathbb{F}_2 \).

**Square root:** complete the factorization.
Recent progresses

Since RSA-155 (512 bits) in 1999, many improvements.

- Very efficient sieving code (Franke, Kleinjung, 2003–).
- Very efficient cofactorization code (Kleinjung, Kruppa).

More recent state of the art, notably for linear algebra:

- Use block Wiedemann algorithm (BW), at separate locations.
- Use computer grids idle time to do linear algebra.
- Use sequences of unbalanced length in BW.
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Asymptotic analysis of NFS gives formulae for:

- asymptotic optimal value for \( \text{deg } f \) (for an \( n \)-bit number).
- asymptotic optimal value for the coefficient sizes.

Trivial “base-\( m \)” approach:

- Choose the degree \( d \). Choose an integer \( m \approx N^{1/(d+1)} \);
- Write \( N \) in base \( m \): \( N = f_d m^d + f_{d-1} m^{d-1} + \cdots + f_0 \).
- Pick \( f = f_d X^d + \cdots + f_0 \) and \( g = X - m \).

We have an immense freedom in the choice of \( m \) \( \implies \) can do better.
Polynomial selection algorithms

Algorithms aim at polynomial pairs \((f, g)\) s.t. \(F(a, b) = b^d f(a/b)\):

- is comparatively small over the sieving range.
- is often smooth \((f\) with many roots mod small \(p\)).

Several relevant algorithms:

- Murphy (1999): rotation and root sieve: \((f, g) \leadsto (f + \lambda g, g)\).

\texttt{Cado-nfs} has a \texttt{polyselect} program implementing this.

- polynomial root finding mod small \(p\);
- knapsack-like problem solving;
- sieving for good \(\lambda\); could use GPUs.
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Sieving: a very old tool

In order to find \((a, b)\) pairs for which \(F(a, b)\) is smooth:

- For all small primes \(p\) (or prime powers);

Note: NFS computation time is mostly spent on sieving.
In order to find \((a, b)\) pairs for which \(F(a, b)\) is smooth:

- For all small primes \(p\) (or prime powers) ;
- for all roots \(r\) of \(f\) mod \(p\), pick \((a_0, b_0)\) s.t. \(a_0 \equiv rb_0 \mod p\) ;

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In order to find \((a, b)\) pairs for which \(F(a, b)\) is smooth:

- For all small primes \(p\) (or prime powers) ;
- for all roots \(r\) of \(f \mod p\), pick \((a_0, b_0)\) s.t. \(a_0 \equiv rb_0 \mod p\) ;
- for all \((u, v)\), mark \((a_0 + pu, b_0 + pv)\) as being divisible by \(p\).

Keep \((a, b)\) pairs which have been marked most.

Do this on both sides \((f \text{ and } g)\). Deciding in which order is subtle.

**Note:** NFS computation time is mostly spent on sieving.
Sieving: describing work

**Lemma.** For coprime \((a, b)\),

\[
\nu_p(F(a, b)) \geq 1 \text{ iff } (a : b) \text{ is a zero of } F \text{ in } \mathbb{P}^1(\mathbb{F}_p).
\]

Example: \(f = 3x^2 + x + 1\).
\[
F(a, b) = 3a^2 + ab + b^2 \equiv 0 \mod 3 \text{ if either:}
\]

- \((a : b) = (2 : 1)\) in \(\mathbb{P}^1(\mathbb{F}_3)\): IOW, \(a - 2b \equiv 0 \mod 3\).
- \((a : b) = (1 : 0)\) in \(\mathbb{P}^1(\mathbb{F}_3)\): IOW, \(b \equiv 0 \mod 3\): “projective”.

More generally, \((a, b)\)’s such that \(\nu_p(F(a, b)) \geq k\) can be described as a set of points in \(\mathbb{P}^1(\mathbb{Z}/p^\ell\mathbb{Z})\).

Starting point of sieving: compute the factor bases (both sides)

- Set of \((p^\ell, r)\), where \(r < 2p^\ell\) encodes a point in \(\mathbb{P}^1(\mathbb{Z}/p^\ell\mathbb{Z})\).
- Algebraic side harder than rational, but done offline anyway.
  - root finding mod \(p\);
  - handle projective roots;
  - handle powers. Some guaranteed headaches.
Typical problems with sieving

There are several practical shortcomings.

- The \((a, b)\) space to be explored is large, but predicting in advance the yield for a range of \((a, b)\) pairs is hard;
- The yield drops as \((a, b)\) grow;
- \(\Rightarrow\) diminishing returns.

Lattice sieving to the rescue.
Old idea (1993), but superiority demonstrated only after 2000.
Lattice sieving

“special-\(q\)”: prime ideal \(q = \langle q, \alpha - r \rangle\).

How do we describe the set of pairs \((a, b)\) such that \(q \mid (a - b\alpha)\)?

Answer: points in the lattice \(\mathcal{L} = \langle e_0 = (r, 1), \ e_1 = (q, 0) \rangle\).

We would like to examine e.g. \(2^{31}\) of these points. Which ones?

- **Bad idea:** \(\{(a, b) = ie_0 + je_1\} \) for \((i, j) \in [−2^{16}, 2^{16}] \times [0, 2^{15}]\). \(a\) gets then too large: \(\approx q \times 2^{15}\).

- **Better:** reduced basis \((e'_0, e'_1)\) and \((i, j)\) in the same range.
  If the reduced basis is nice, we expect \(a \approx b \approx 2^{16} \sqrt{q}\). 

\[\text{Diagram of lattice points}\]
Lattice sieving

“special-$q$”: prime ideal $q = \langle q, \alpha - r \rangle$.

How do we describe the set of pairs $(a, b)$ such that $q \mid (a - b\alpha)$? Answer: points in the lattice $\mathcal{L} = \langle e_0 = (r, 1), \ e_1 = (q, 0) \rangle$.

We would like to examine e.g. $2^{31}$ of these points. Which ones?

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- Better: reduced basis $(e'_0, e'_1)$ and $(i, j)$ in the same range. If the reduced basis is nice, we expect $a \approx b \approx 2^{16} \sqrt{q}$.

Benefits

- A factor of $q$ is forced in the norm ;
- for $q$’s of comparable size, we have comparable yields ;
- immense choice of special-$q$’s ;
- smaller sieve areas.
Lattice sieving: how do we sieve?

Given a special-q and \( \begin{pmatrix} e'_0 \\ e'_1 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix} \), we consider the lattice

\[ L_q = \{ (a, b) = ie'_0 + je'_1 \} . \]

All work is done on the \((i, j)\) plane. A rectangle \( R_{(i,j)} \) is fixed.

The workplan for sieving for this special \( q \) is:

- Describe locations to sieve in the \((i, j)\) plane.
- Sieve “small” factor base primes.
- Sieve “large” factor base primes.
- Do this for both sides.
- Locations which have been marked most need to be factored.
Sieve locations in the \((i, j)\) plane.

Let \(p\) be a prime (power) coprime to \(q\). We have a homography:

\[
h_q : \begin{cases} 
\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}) & \rightarrow \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}), \\
(i : j) & \mapsto (a : b) = (ia_0 + ja_1 : ib_0 + jb_1).
\end{cases}
\]

Starting from a description \(S_p\) of the \((a, b)\) sieve locations:

\[
\{(i, j), p \mid F(a, b)\} = \{(i, j), (a : b) \in S_p \subset \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})\},
\]

\[
= \{(i, j), h_q(i : j) \in S_p\},
\]

\[
= \{(i, j), (i : j) \in h_q^{-1}S_p\}.
\]

- This change of basis must be redone for each \(q\).
- relatively cheap because independent of the sieve area size.
- Need to precompute preinverses of factor base primes.
Fine points of sieving

For a given $q$, explore some $\mathcal{R}_{(i,j)}$ of size e.g. $2^{31}$.

- Divide into areas matching L1 cache size (64kb typically), to be processed one by one.
- Small primes hit often: once per row.
- Larger primes hit rarely. Rather maintain a “schedule” list to circumvent cache misses: “bucket sieving”.
- Use multithreading.

CADO-NFS implements this in las.

- Hot spots in assembly; Use vector instructions when relevant;
- Optimize some data structures to reduce memory footprint;
- Strive to eliminate badly predictable branches;
- POSIX threads;
- Factoring good $(a, b)$’s: Use $p \pm 1$ and special-purpose ECM.
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The output of the sieve process is a set of relations. These undergo:

- Filtering: making a small relation set from a large one;
- After filtering, linear system solving.

Algorithmically, nothing very new in filtering since Cavallar (2000).

Implementation in CADO-NFS:

- Hash tables all over the place;
- Minimum spanning trees to help decision;
- Has supported MPI distribution at some point;

Does the job so far.
Must **combine** relations so that they consist of only squares. This rewrites as a **linear system**. (everything reduces to lin. alg. !)

- **matrix** $M$: a relation appears in each row. Coefficients are multiplicities of prime factors (and ideals). **Most are zero.**
- A **vector** $v$ such that

\[
 vM = 0 \pmod{2}
\]

indicates which relations to combine in order to obtain only squares (even multiplicities).

Equivalently, we rephrase this as a linear system $Mv = 0$ (transposing $M$).

Note: linear algebra mod 2 differs much from linear algebra over $\mathbb{C}$.
$\mathbb{F}_2$ is exact, and positive characteristic

(some PDE example) (a factoring matrix)
Linear algebra

We have an $N \times N$ matrix $M$. We want to solve $Mw = 0$. The matrix $M$ is large, (very) sparse, and defined over $\mathbb{F}_2$. Because of sparsity, we want a black box algorithm.

There are several sparse linear algebra algorithms suitable for $\mathbb{F}_2$:

- Lanczos;
- Wiedemann; others.

These early suggestions are unsuitable. Bit arithmetic: slow. Also, failure probability $1/\#\mathbb{F}_2 = 1/2$ is not so tempting...
Block algorithms

Block algorithms apply the black box to e.g. $n = 64$ vectors at a time. ($n$ is prescribed by the hardware)

- Block Lanczos (BL). $\frac{2^N}{n^{0.76}}$ black box applications;
- Block Wiedemann (BW). $\frac{3^N}{nn'}$, $n'$ times ($n'$ small).

BL is appealing if one has a large cluster.

BW is preferred since it offers distribution opportunities.
Block Wiedemann: workplan

- Initial setup. Choose starting blocks of vectors $x$ and $y$.
- Sequence computation. Want $L$ first terms of the sequence:

\[ a_i = x^T M^k y. \]

- Computing one term after another, this boils down to our black box $v \mapsto Mv$.
- This computation can be split into several independent parts (which all know $M$).

- Compute some sort of minimal polynomial.
- Build solution as:

\[ v = \sum_{k=0}^{\deg f} M^k y f_k. \]

- Again, this uses the black box.
- Can be split into many independent parts (which all know $M$).
The matrix $M$ itself is soon out of reach for core storage.

- 2005: kilobit SNFS: 64M rows/cols, 10G non-zero coeffs. About 30GB.
- 2010: 768b GNFS: 192M rows/cols, 27G non-zero coeffs. About 75GB.

Computing $M \times v$ is also a lot of work. Try to use many processors if possible.

This is a classical HPC concern.

- Split the matrix into equal parts.
- Exploit high-bandwidth channels: shared memory, infiniband network.
Features of the **Cado-NFS BW code**

**Cado-NFS** has a complete BW implementation.

**Sequence computation:**

- POSIX threads;
- MPI – implementation agnostic. Some optimized collectives;
- Some kind of “sparse binary BLAS” used. Assembly;
- (Stem of) capability to switch to other base field;
- Mostly C, some C++. Wrapper script in Perl.

**Minimal polynomial computation using a quasi-linear algorithm.**

- recursive structure;
- arithmetic on matrices of polynomials over $\mathbb{F}_2$.
- very old code, needs rework.
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The square root step

Our congruence of squares actually comes as:
\[(a_1 - b_1 m) \times \cdots \times (a_k - b_k m) \equiv \phi((a_1 - b_1 \alpha) \times \cdots \times (a_k - b_k \alpha)) \mod N.\]

Both sides are known to factor with even multiplicities: they are squares.
BUT computing the square root is in fact non trivial (esp. on algebraic side).

\textbf{CADO-NFS} implements quasi-linear algorithms for this

- Newton lifting.
- Arithmetic modulo fixed degree polynomials.
- Suitable for current records.
- Alternative algorithm (waives a number theoretic assumption):
  - Explicit CRT.
  - Can be distributed with MPI.

There exists a more advanced square root algorithm for this step (Montgomery), but it needs more software support.
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Conclusion and further work

Many points would be interesting to improve.

- Polyselect with GPUs (but msieve does this already).
- Lattice siever needs cleanup, and some obvious improvements.
- Filtering currently can’t handle record sizes.
- Linear algebra sparse BLAS can be improved.
- Linear algebra minimal polynomial step must be reworked.
- The whole chain could be adapted to discrete log computation.