M2 internship proposal 2023–2024

Title: Sparse interpolation of rational functions

Topics: symbolic computation, high performance computing

Address

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Context

The MAX team is searching for PhD candidates on the themes of the ANR "NODE" project. The present M2 internship proposal allows applicants to familiarize themselves with these themes. Upon successful completion of the internship, there will be an opportunity to pursue with a PhD. The ANR NODE project provides funding for two PhD grants.

Description

Large-scale symbolic computations lead to the manipulation of very large mathematical expressions. Often, such expressions can be regarded as so-called *sparse* polynomials or rational functions in suitable variables. For instance, a sparse polynomial of degree 100 in x, y, and z is a polynomial $\sum_{0 \leqslant i,j,k \leqslant 100} P_{i,j,k}\, x^i y^j z^k$, for which the number of non-zero terms $P_{i,j,k} x^i y^j z^k$ is significantly lower than 101^3.

One typical motivating example of a sparse polynomial is the determinant Δ of the symbolic matrix

$$M := \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}.$$

Indeed, Δ is a polynomial of degree n in n^2 variables, with $n! \ll (n+1)^{n^2}$ terms. The entries of the matrix inverse M^{-1} of M are typical examples of sparse rational functions.

One notorious enemy in symbolic computation is *intermediate expression swell*. For instance, when computing Δ symbolically, using Gaussian elimination, the intermediate expressions tend to become much larger than the final result. A magic way to defeat this enemy is *sparse interpolation*. The idea is to evaluate the determinant Δ for a sufficient number of well chosen values of the variables $(a_{i,j})_{1 \leqslant i,j \leqslant n}$ and then to reconstruct Δ from these evaluations.

The idea of sparse interpolation for polynomials goes back a long while [1, 6, 7]. Various methods have been proposed to generalize it to rational functions [2, 5]. Recently, particularly efficient methods for these tasks have been proposed [3, 4], but not yet implemented. The purpose of this internship is to learn and implement these new ideas and further improve them, if possible. If successful, then this may lead to significantly faster future implementations of many basic tasks in computer algebra.

Bibliography

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