PhD proposal

Title: Computing transseries solutions to differential equations

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M2 students majoring in computer science or mathematics may apply. Basic familiarity with computer algebra is required. Knowledge about differential algebra is appreciated, but optional.

Description

How to compute exactly and reliably with special functions in computer algebra systems? One strategy is to adopt a local approach and systematically represent such functions using power series around some non-singular point. For instance, the function $f(z) := \exp(\sin(z))$ is the unique power series solution to the system of equations $f'(z) = c(z) f(z)$, $c'(z) = -s(z)$, $s'(z) = c(z)$, with initial conditions $f(0) = 1$, $c(0) = 1$, $s(0) = 0$. Assuming this kind of representation, how to check equalities like $s(z)^2 + c(z)^2 = 1$?

Clearly, it suffices to have an algorithm for checking whether a given expression represent the zero function. When all power series are given as solutions of explicit differential equations with explicit initial conditions, several algorithms have been proposed for this zero test problem [2, 7, 6, 5]. One first challenge is to implement one or more of these algorithms and investigate possible improvements.

A second challenge concerns the computation of the generic power series solution of a system of ordinary differential equations if the initial conditions are not or only partially specified. This requires an extension of the theory to the case when the coefficients of the power series involve parameters. For instance, setting $\delta := z \partial / \partial z$, the general solution to the equation $(\delta f)^2 - f \delta^2 f = 0$ is $f = c z^n$, where c is an arbitrary constant and $n \in \mathbb{N}$ an arbitrary non-negative integer.

The main endgoal of the thesis is to further generalize this theory to so-called transseries instead of ordinary power series [3, 4, 1]. A transseries is a generalized power series that is allowed to recursively involve exponentials and logarithms. For instance, $\int \mathrm{e}^{\mathrm{e}^x} = \mathrm{e}^{\mathrm{e}^x - x} + \mathrm{e}^{\mathrm{e}^x - 2x} + 2\,\mathrm{e}^{\mathrm{e}^x - 3x} + \cdots$ and $W(x) := (x\,\mathrm{e}^x)^{\text{inverse}} = \log x - \log \log x + \frac{\log \log x}{\log x} + \cdots$ are transseries at infinity $x \to \infty$. Transseries naturally arise when studying the asymptotic behavior of solutions to ordinary differential equations. Ultimately, we would like to have efficient zero tests for transseries and algorithms to find transseries solutions of differential equations.

Methodology

The thesis will start with a study of the existing literature. Next increasingly difficult cases of zero testing and resolution of differential equations will be considered. Depending on her or his profile, the PhD student may choose to put greater emphasize on the theoretical or practical aspects of the problem. Software implementations will be open source and could done in C++, JULIA, MATHEMAGIX, or a mixture of these languages.

Expected results

The theoretical part of the research should at least lead to a new zero test for differentially algebraic transseries. In addition, we hope that progress will be made on the computation of power series and transseries solutions to differential equations.

The algorithms will be tested at least on toy implementations. Depending on her or his profile, the PhD student may choose to develop a more advanced implementation that can be distributed as a library.

The new results should be published in prominent journals or conference proceedings of the area, such as JSC, AAECC, ISSAC, etc.

Bibliography

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